Guide 2 Exercise 4

Given the function $f(x, y) = ax^2y + bxy$, find the values of the parameters a and b such that the derivative at the point $P = (1, 1)$ is maximal in the direction of the vector $\mathbf{v} = (3, 4)$ and equals $f'_{\text{Max}}(1, 1) = 15$.

Solution

We calculate the partial derivatives:

$$f'_x = 2axy + by$$

$$f'_y = ax^2 + bx$$

Evaluating at the point:

$$f'_x = 2a + b$$

$$f'_y = a + b$$

The directional derivative is maximal when the direction vector is in the same direction and sense as the gradient vector. The value of the maximal directional derivative is $\|\nabla f(x_0; y_0)\|$.

In this case, the gradient vector is:

$$\nabla f = (2a + b, a + b)$$

On the other hand, the direction vector has the following norm:

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Therefore, the associated unit vector is $(3/5; 4/5)$. Furthermore, the exercise requests that the maximal directional derivative be equal to 15:

$$D'_{\mathbf{u}}z(x_0; y_0) = \nabla f(x_0; y_0)(\hat{u}_1 \hat{u}_2)$$

We calculate the dot product and equate to 15.

$$(2a + b)3/5 + (a + b)4/5 = 15$$

$$6a + 3b + 4a + 4b = 75$$

$$10a + 7b = 75$$

Finally, it is necessary that the gradient vector has the same direction and sense as the unit vector, meaning that one is a scalar multiple of the other:

$$\nabla f K = (3/5; 4/5)$$

$$(2a + b)K = 3/5$$

$$(a + b)K = 4/5$$

$$\frac{4/5}{a + b} = \frac{3/5}{2a + b}$$

$$8a/5 + 4b/5 = 3a/5 + 3b/5$$

$$-5a = b$$

We replace this in the relation we had above:

$$10a + 7 * (-5a) = 75$$

$$a = -3$$

Then with this, we obtain the value of b:

$$b = 15$$